V-SUPERMAGIC *H*-DECOMPOSABLE DIGRAPHS

G. MARIMUTHU, B. JOHNSON and M. BALAKRISHNAN

Department of Mathematics
The Madura College
Madurai, Tamil Nadu, India
E-mail: yellowmuthu@yahoo.com

Department of Mathematics
The American College
Madurai, Tamil Nadu, India
E-mail: jj_311280@yahoo.co.in

Department of Mathematics
Arulmigu Kalasalingam College of Arts and Science
Krishnankoil, Tamil Nadu, India
E-mail: balki_ajc@rediffmail.com

Abstract

An out-magic labeling f of a digraph D is said to be a V-super vertex out-magic labeling if $f(V(D)) = \{1, 2, 3, \ldots, |V(D)|\}$. A decomposition of a digraph D into isomorphic copies of a graph H is H-magic if there is a bijection f from $V(D) \cup A(D)$ to the consecutive integers $1, 2, 3, \ldots, |V(D)| + |A(D)|$ such that for every copy H in the decomposition, $\sum_{v \in A(H)} f(v) + \sum_{(u,\, v) \in A(H)} f((u,\, v))$ is constant. An H-magic labeling f is called V-super H-magic if it has an additional property that $f(V(D)) = \{1, 2, \ldots, |V(D)|\}$. In this paper, we introduce the concept of V-super vertex out-magic labeling and V-super H-magic labeling in digraphs. Further, we investigate the existence of these labelings in generalized de Bruijn digraphs.

1. Introduction

In this paper, we consider only finite and simple directed graphs which admit loops but no multiple arcs. The vertex and arc sets of a digraph D are

denoted by $V(D)$ and $A(D)$ respectively and we let $|V(D)| = p$ and $|A(D)| = q$. For graph theoretic notations, we follow [2, 12]. A labeling of a graph G is a mapping that carries a set of graph elements, usually vertices and or edges into a set of numbers (integers). Many kinds of labeling have been studied and an excellent survey of graph labeling can be found in [5]. The notion of a V-super vertex magic total labeling of graphs was introduced by MacDougall, Miller and Sugeng [9] as in the name of super vertex magic labeling and it was renamed as V-super vertex magic labeling by Alison M. Marr and W. D. Wallis in [12] by refering the article E-super vertex magic labelings of graphs by G. Marimuthu and M. Balakrishnan [11]. A vertex magic total labeling is said to be E-super vertex magic labeling if the smallest numbers are assigned to the edges.

The notion of H-super magic labeling in undirected graph was first introduced and studied by Gutierrez and Llado [6] in 2005. A total labeling $f : V(G) \bigcup E(G)$ to the consecutive integers $1, 2, 3, \ldots p + q$ is called an H-magic labeling of G if there exists a positive integer k (called magic constant) such that for every subgraph H' of G isomorphic to H and $\sum_{v \in V(H')} f(v)$

$+ \sum_{e \in E(H')} f(e) = k.$ A graph G that admits such a labeling is called H-magic graph. G is said to be H-supermagic if $f(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$. They [6] proved that some classes of connected graphs are H-super magic. Llado and Moragas [8] studied the cycle-magic and cyclic-super magic behavior of several classes of connected graphs and they gave several families of C_r-magic graphs for each $r \geq 3$. Several authors [10, 13, 14] studied H-magic labeling on different class of graphs. Toru Kojima [17] studied the C_4-super magic labeling of the Cartesian product of paths and graphs, which are called H-super magic if the smallest labels are assigned to the vertices. In 2012, Inayah, Llado and Moragas [7] studied the magic and antimagic H-decomposition of the complete bipartite graphs and Zhihe Liang [18] studied cycle-supermagic decomposition of complete multipartite graphs which enable us to extend these labelings in digraphs.

Any spanning subdigraph of a digraph D is referred to as a factor of D. A digraph D is said to be factorable into the factors $D_1, D_2, D_3, \ldots, D_t$ such

that these factors are pairwise arc disjoint and $\bigcup_{i=1}^{t} A(D_i) = A(D)$. If D is factorable into $D_1, D_2, D_3, \ldots, D_t$, we represent this by $D_1 \oplus D_2 \oplus D_3 \oplus \ldots \oplus D_t$ which is called a factorization of D. In particular, if D is factorable into $D_1, D_2, \ldots, D_t$, such that $D_i \cong H$ for some digraph H, we say that D has an isomorphic factorization into the factor H. We write $D = H_1 \oplus H_2$ if D is the arc-disjoint union of the subdigraphs H_1 and H_2. If $D = H_1 \oplus H_2 \oplus \ldots \oplus H_k$, where $H_1 \cong H_2 \cong \ldots \cong H_k \cong H$, then the digraph D can be decomposed into subdigraphs isomorphic to H and we say that D is H-decomposable.

The de Bruijn digraph defined in [3] has been noted as an interconnection network for massively parallel computers because of its good properties such as small diameter, high connectivity and easy routing (see [1]). The generalized de Bruijn digraph $G_B(n, d)$ is defined in [1, 3] by the congruence equations as follows:

$$V(G_B(n, d)) = \{0, 1, 2, \ldots, n-1\} \text{ and}$$

$$A(G_B(n, d)) = \{(x, y) : y \equiv dx + i \,(\mathrm{mod}\, n), 0 \leq x \leq d-1\}.$$

Though different kinds of labelings were studied by many authors and many conjectures were made for different subclasses of graphs and digraphs, the labeling of some well-known digraphs, namely generalized de Bruijn digraphs has not yet been investigated.

Now, we introduce some definitions and provide some existing results in the literature which are needed to formulate this article.

Definition 1.1 [12]**.** An out-magic total labeling of a digraph D is a bijection f from $V(D) \cup A(D)$ to the consecutive integers $1, 2, 3, \ldots, p+q$ with the property that for every vertex $u \in V(D)$, sum at u, $f(u) + \sum_{(u, v) \in A(D)} f((u, v)) = k$ and k is called the magic constant. A digraph which admits vertex out-magic total labeling is called vertex out-magic digraph.

Definition 1.2 [12]**.** An in-magic total labeling of a digraph D is a bijection f from $V(D) \cup A(D)$ to the consecutive integers $1, 2, 3, \ldots, p+q$ with the property that for every vertex $u \in V(D)$, sum at u, $f(u) + \sum_{(u, v) \in A(D)} f((v, u)) = k$, and k is called the magic constant. A

digraph which admits vertex in-magic total labeling is called vertex in-magic digraph.

Definition 1.3. A out-magic total labeling is said to be a *V*-super vertex out-magic labeling if $f(V(D)) = \{1, 2, 3, \ldots, p\}$. A digraph which admits *V*-super vertex out-magic total labeling is called *V*-super vertex out-magic digraph.

Definition 1.4. A in-magic total labeling is said to be a *V*-super vertex in-magic labeling if $f(V(D)) = \{1, 2, 3, \ldots, p\}$. A digraph which admits *V*-super vertex in-magic total labeling is called *V*-super vertex in-magic digraph.

In our terminology, we call *V*-super vertex out-magic labeling as *V*-super vertex magic labeling and *V*-super vertex out-magic digraphs as *V*-super vertex magic digraphs.

Definition 1.5 [12]**.** An arc-magic labeling on a digraph *D* is a bijective map *f* from $V(D) \bigcup A(D)$ to the consecutive integers $1, 2, 3, \ldots, p+q$ in which the sum $f((x, y)) + f(y)$ is constant for every arc $(x, y) \in A(D)$. An arc-magic labeling is said to be *V*-super arc-magic if $f(V(D)) = \{1, 2, 3, \ldots, p\}$.

Definition 1.6. An *H*-magic labeling in a *H*-decomposable digraph *D* is a bijection *f* from $V(D) \bigcup A(D)$ to the consecutive integers $1, 2, 3, \ldots, p+q$ such that the sum of labels of arcs and vertices of each copy *H* in the decomposition is constant. An *H*-decomposible digraph which admits *H*-magic labeling is said to be magic *H*-decomposable digraph. An *H*-magic labeling *f* is called a *V*-super *H*-magic if it has an additional property that $f(V(D)) = \{1, 2, \ldots, p\}$. An *H*-decomposible digraph *D* which admits *V*-super *H*-magic labeling is said to be *V*-supermagic *H*-decomposable digraph.

In a digraph, the weight of each arc is defined as the sum of the arc-label and the label of the vertex incident from the arc. If the weight of each arc in a digraph *D* is equal, then we call it as a arc-magic digraph. That is, for each arc the sum of the labels in the sub digraph induced by the arc and the vertex label of the vertex incident from the arc is the same. For, if $H \approx \vec{K}_2$, then *H*-(super) magic digraph is also called (super) arc-magic digraph.

Definition 1.7 [15]**.** A digraph *D* is a cycle-rooted tree if and only if *D* is weakly connected and every vertex of *D* has indegree one.

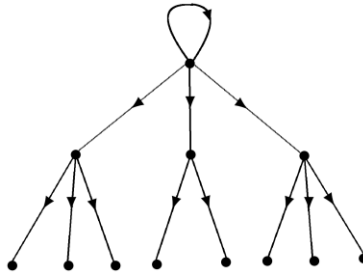**Definition 1.8** [15]**.** A cycle-rooted tree is said to be loop-rooted tree when the root cycle is a loop.

Figure 1. An example of a loop-rooted tree.

Theorem 1.9 [15]**.** *Let* $g = \gcd(d - 1, n)$. *Then* $G_B(n, d)$ *is factorable into loop-rooted trees if and only if* $g = 1$.

Theorem 1.10 [15]**.** *Let d and n be integers satisfying* $(d - 1)|(n - 1)$ *and* p *be any integer satisfying* $d^p \leq n < d^{p+1}$. *Then,* $G_B(n, d)$ *contains a complete d-ary loop-rooted tree of the height p with the root at each loop.*

Theorem 1.11 [4]**.** *For given odd integers* $p > 1$ *and* $q > 1$, *there exists a* $p \times q$ *magic rectangle M.*

Theorem 1.12 [16]**.** *For* $n > 1$ *an odd integer*; *there exists a magic* $(3 \times n)$-*rectangle R such that one row of R contains all the integers from* $n + 1$ *to* $2n - 1$; *with the exception of* $\dfrac{3n + 1}{2}$.

In Section 2, we discuss *V*-super vertex magic labeling in generalized de Bruijn digraphs. Section 3 contains a discussion of *V*-super *factor*-magic labeling of digraphs. In Section 4, *V*-super loop-rooted tree-magic labeling in generalized de Bruijn digraphs are investigated. Section 6 contains conclusion and some open problems.

2. *V*-super Vertex Magic Digraphs

In this section, we provide some basic properties of *V*-super vertex magic labeling. Using these properties, we determine whether a generalized de Bruijn digraph admits *V*-super vertex magic labeling or not.

Now, we are going to determine the magic constant for *V*-super vertex magic digraphs.

Theorem 2.1. *If a non-trival digraph D is V-super vertex magic, then the magic constant k is given by* $k = q + \dfrac{p+1}{2} + \dfrac{q(q+1)}{2p}$.

Proof. Let f be a V-super vertex magic total labeling of a digraph D with magic constant k. Then we have: $f(V(D)) = \{1, 2, 3, \ldots, p\}$ and $f(A(D)) = \{p+1, p+2, p+3, \ldots, p+q\}$ such that $k = f(u) + \sum_{(u,v) \in A(D)} f((u,v))$ for all $u \in V(D)$. Now, it follows that:

$$pk = \sum_{u \in V(D)} f(u) + \sum_{(u,\,v) \in A(D)} f((u,\,v))$$

$$= 1 + 2 + 3 + \ldots + p + p + 1 + \ldots + p + q.$$

This implies that, $k = \dfrac{(p+q)(p+q+1)}{2p} = q + \dfrac{p+1}{2} + \dfrac{q(q+1)}{2p}$.   □

Corollary 2.2. *If a digraph D is connected and admits V-super vertex magic labeling with magic constant k, then* $k \geq 2p - 1$.

Proof. If a digraph D is connected, then $q \geq p - 1$. By Theorem 2.1, we obtain $k \geq p - 1 + \dfrac{p+1}{2} + \dfrac{p(p-1)}{2p} = 2p - 1$.

The following theorem gives a necessary and sufficient condition for a digraph to be a *V*-super vertex magic. This theorem is helpful in deciding whether a particular digraph has a *V*-super vertex magic labeling.

Theorem 2.3. *Let D be a digraph and f be a bijection from $A(D)$ onto $\{p+1, p+2, \ldots, p+q\}$. Then g can be extended to a V-super vertex magic labeling if and only if $\{w(u) = \sum_{(u,v) \in A(D)} f((u,v)) : u \in V(D)\}$ consists of p consecutive integers.*

Proof. Assume that $\{w(v) : v \in V(D)\}$ consists of p consecutive integers. Let $t = \min\{w(v) : v \in V(D)\}$. Define $g : V(D) \bigcup A(D)$ to $\{1, 2, \ldots, p+q\}$ as $g((u, v)) = f((u, v))$, for $(u, v) \in A(D)$ and $g(v) = p + t - w(v)$. Then we obtain $g(V(D)) = \{1, 2, \ldots, p\}$ and $g(A(D)) = \{p+1, p+2, \ldots, p+q\}$. Hence g is a *V*-super vertex magic total labeling with magic constant $k = p + t$.

Conversely, suppose that f can be extended to a *V*-super vertex magic total labeling g of D with magic constant k. Let $t = \min\{w(v) : v \in V(D)\}$. Since for every $v \in V(D)$, $g(v) + w(v) = k$, we have $w(v) = k - g(v)$. Thus, $\{w(v) : v \in V(D)\} = \{k-1, k-2, \ldots, k-p\} = \{t, t+1, \ldots, p+q\}$, where $t = k - p$. Hence the theorem follows.   □

The following theorem is an immediate consequence of Theorem 2.1.

Theorem 2.4. *If $G_B(n, d)$ is V-super vertex magic, then the magic constant k is given by* $k = nd + \dfrac{n+1}{2} + \dfrac{d(nd+1)}{2}$.

Proof. By the definition of $G_B(n, d)$, $p = |V(D)| = n$ and $q = |A(D)| = nd$. By Theorem 2.1, we have, $k = nd + \dfrac{n+1}{2} + \dfrac{d(nd+1)}{2}$.   □

It is not easy to find a *V*-super vertex magic generalized de Bruijn digraph $G_B(n, d)$. The total labeling assign integers $1, 2, 3, \ldots, p + q = n + nd$ to the vertices and arcs. The total sum of the labeling is $\dfrac{n(d+1)(n(d+1)+1)}{2}$. If the labeling is magic, then the magic constant is $\dfrac{n(d+1)(n(d+1)+1)}{2n} = \dfrac{(d+1)(nd+n+1)}{2}$ to be an integer either d is odd or n is odd. So, we divide them into three categories.

1. n is even and d is even.

2. d is odd.

3. n is odd and d is even.

Theorem 2.5. *The digraph $G_B(n, d)$ is not V-super vertex magic, if both n and d are even.*

Proof. The result immediately follows from the Theorem 2.4.   □

Theorem 2.6. *The digraph $G_B(n, d)$ is V-super vertex magic, when d is odd.*

Proof. By the definition of $G_B(n, d)$, $V(D) = \{0, 1, 2, n-1\}$ and $A(D) = \{f(x, y)/y \equiv dx + i \pmod{n}, i = 0, 1, \ldots, d-1\}$. Define a bijection

$f : V(D) \cup A(D)$ to $\{1, 2, \ldots, n(d + 1)\}$ by $f(v) = v + 1$, for all $v \in V(G_B(n, d))$.

The arc set of $G_B(n, d)$ can be written as $A(G_B(n, d)) = \bigcup_{k=0}^{n-1} A_k$, where

$A_k = \{f(k, x) : \ x \equiv dk + i (\operatorname{mod} n)$ and $i = 0, 1, \ldots, d - 1\}$. The labeling for the arcs of $G_B(n, d)$ is given in Table 1. Using Table 1, the sum at the vertex 0 can be calculated as follows:

$$k = 1 + 2n + 2n + 1 + 4n + 4n + 1 + \ldots + n(d - 1) + 1 + n(d + 1)$$

$$= \frac{d + 1}{2} + 4n\left(1 + 2 + 3 + \ldots + \frac{d - 1}{2}\right)$$

$$= \frac{d + 1 + n(d - 1)(d + 1) + 2n(d + 1)}{2}$$

$$= \frac{(d + 1)(1 + nd - n + 2n)}{2}$$

$$= \frac{(d + 1)(nd + n + 1)}{2}.$$

In a similar way, we can show that the sum at every vertex is $\frac{(d + 1)(nd + n + 1)}{2}$.   □

Theorem 2.7. *The digraph* $G_B(n, d)$ *is V-super vertex magic, when n is odd and d is even.*

Table 1. Arc labels.

A_k	Arc labels for arcs in A_k						
A_0	$2n$	$2n+1$	$4n$	$4n+1$	…	$n(d-1)+1$	$n(d+1)$
A_1	$2n-1$	$2n+2$	$4n-1$	$4n+2$	…	$n(d-1)+2$	$n(d+1)-1$
A_2	$2n-2$	$2n+3$	$4n-2$	$4n+3$	…	$n(d-1)+3$	$n(d+1)-2$
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
A_{n-2}	$n+2$	$3n-1$	$3n+2$	$5n-1$	…	$nd-1$	$nd+2$
A_{n-1}	$n+1$	$3n$	$3n+1$	$5n$	…	nd	$nd+1$

Table 2. Arc labels.

A_k	Arc labels for arcs in A_k						
A_0	$\dfrac{3n+1}{2}$	$2n+1$	$4n$	$4n+1$	…	nd	$nd+1$
A_1	$2n$	$2n+2$	$4n-1$	$4n+2$	…	$nd-1$	$nd+2$
A_2	$\dfrac{3n-1}{2}$	$2n+3$	$4n-2$	$4n+3$	…	$nd-2$	$nd+3$
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
A_{n-2}	$\dfrac{3n+3}{2}$	$3n-1$	$3n+2$	$5n-1$	…	$n(d-1)+2$	$n(d+1)-1$
A_{n-1}	$n+1$	$3n$	$3n+1$	$5n$	…	$n(d-1)+1$	$n(d+1)$

Proof. Let $V(G_B(n,d)) = \{0, 1, 2, …, n-1\}$ be the vertex set. The arc set of $G_B(n, d)$ can be written as $A(G_B(n, d)) = \bigcup_{k=0}^{n-1} A_k,$ where $A_k = \{f(k, x) : x \equiv dk + i(\mathrm{mod}\, n),\ i = 0, 1, …, d-1\}$. The labeling of the arcs of $G_B(n, d)$ is given in Table 2. In Table 2, at each vertex $k,\ k = 0, 1, 2, …, n-1,$ the sum of the labels in the first and second columns

are the integers $\dfrac{7n+3}{2}$, $4n+2$, $\dfrac{7n+5}{2}$, ..., $\dfrac{9n+1}{2}$, $4n+1$. After rearranging,

we obtain the consecutive integers $\dfrac{7n+3}{2}$, $\dfrac{7n+5}{2}$, $\dfrac{7n+7}{2}$, ..., $4n$, $4n+1$, $4n+2$,

..., $\dfrac{9n-1}{2}$, $\dfrac{9n+12}{2}$. At each vertex k, $k = 0, 1, 2, ..., n-1$, the sum of the

next $d-2$ labels in each row is $\dfrac{nd(d+2)-8n+d-2}{2}$. By combining these

numbers at each vertex, we get a sequence of consecutive integers. By Theorem 2.3, this labeling can be extended to a V-super vertex magic labeling.   □

3. V -Supermagic Factor-Decomposable Digraphs

This section will explore the properties of V-supermagic *factor-*decomposable digraphs.

Theorem 3.1. *If a non-trival digraph D admits a V-super factor-magic labeling, then the magic constant k is given by* $k = \dfrac{p(p+1)}{2} + \dfrac{q^2 + 2pq + q}{2h}$,
where h is the number of factors of D.

Proof. Let f be a V-super factor-magic labeling of a digraph D with magic constant k. Then $f(V(D)) = \{1, 2, 3, ..., p\}$ and $f(A(D)) = \{p+1, p+2, p+3, ..., p+q\}$ such that $k = \sum_{v \in V(D')} f(v) + \sum_{(u,v) \in A(D')} f(v)$ for all $u \in V(D')$ and for every factor D' of D. Then,

$$hk = \sum_{v \in A(D)} f(v) + \sum_{(u,v) \in A(D)} f((u, v))$$

$$= h(1 + 2 + 3 + ... + p) + (p + 1 + ... + p + q).$$

That is,

$$k = \frac{p(p+1)}{2} + \frac{(p+q)(p+q+1)}{2h} - \frac{p(p+1)}{2h} = \frac{p(p+1)}{2} + \frac{q^2 + 2pq + q}{2h}. \qquad □$$

If a digraph D is V-supermagic factor-decomposable, then it can be easily seen that the sum of the vertex labels denoted by k_v in each copy remains to be the same. This gives the following result.

Theorem 3.2. *If a non-trivial digraph D is V-supermagic factor-decomposable, then the sum of the arc labels denoted by k_e is constant and it is given by $k_e = \dfrac{q^2 + 2pq + q}{2h}$, where h is the number of factors of D.*

Proof. Assume D has a *V*-super factor-magic labeling f. then by Theorem 3.1, the magic constant is given by $k = \dfrac{p(p+1)}{2} + \dfrac{q^2 + 2pq + q}{2h}$, for every factor of D. Since D is H-decomposable, every subdigraph H' of D which is isomorphic to H in the decomposition is a factor of D. It follows that k_v is constant for every factor in the decomposition of D and also $k = k_v + k_e$. Thus k_e must be a constant. Also

$$hk_e = \sum_{e \in A(D)} f(e) = p + 1 + p + 2 + p + 3 + \ldots + p + q$$

$$= \frac{qp + q(q+1)}{2h}$$

$$= \frac{(p+q)(p+q+1)}{2h} - \frac{p(p+1)}{2h}$$

$$= \frac{q^2 + 2pq + q}{2}.$$

Therefore $k_e = \dfrac{q^2 + 2pq + q}{2}$. □

Now we develop a necessary and sufficient condition for a factorable digraph to be a *V*-supermagic factor-decomposable digraph.

Theorem 3.3. *Let D be a factorable digraph and let f be a bijection from $A(D)$ onto $\{p+1, p+2, \ldots, p+q\}$. Then f can be extended to a V-super factor-magic labeling of D if and only if $k_e = \sum_{e \in A(D)} f(e)$ is constant for every factor D' of D.*

Proof. Let $V(D) = \{v_1, v_2, v_3, \ldots, v_p\}$. Assume that $k_e = \sum_{e \in A(D)} f(e)$ is constant for every factor D' of D. Define $g : V(D) \cup A(D)$ to

$\{1, 2 \ldots, p + q\}$ as $g((u, v)) = f((u, v))$, for every $(u, v) \in A(D)$ and $g(v_i) = i$, for all $i = 1, 2, \ldots, p$. Since D' is a factor of D, $k_v = \sum_{v \in V(D')} f(v) = \dfrac{p(p + 1)}{2}$, for every factor D' of D. Therefore $k_v + k_e = \sum_{v \in V(D')} f(v) + \sum_{e \in A(D')} f(e)$ is a constant for every factor D' of D. Thus g is a V-super factor-magic labeling of D. Conversely, suppose that f can be extended to a V-super factor-magic labeling g of D with constant k. Then $k = \sum_{v \in V(D')} f(v) + \sum_{e \in A(D')} f(e)$ for every factor D' of D. Since D is factorable, $k_v = \sum_{v \in V(D')} f(v) = \dfrac{p(p + 1)}{2}$ and it follows that $k_e = \sum_{e \in A(D)} f(e)$, which is a constant for every factor D' of D.   □

4. *V*-supermagic Loop-rooted Tree-decomposable Generalized de Bruijn Digraphs

In this section, we discuss the concept of *V*-super loop-rooted tree-magic labeling in generalized de Bruijn digraphs. It is not easy for a generalized de Bruijn digraph into a *V*-supermagic loop-rooted tree-decomposable. So we restrict our attention to some special class of generalized de Bruijn digraphs. First, we consider the case $d = 3$ and *n* is odd.

Theorem 4.1. *The digraph $G_B(n, 3)$ is V-supermagic loop-rooted tree-decomposable, when n is odd.*

Proof. Since $\gcd(n, 2) = 1$, by Theorem 1.9, we can easily decompose $G_B(n, 3)$ into three arc disjoint sub digraphs. In order to decompose the digraph into loop-rooted trees, the following cases are needed.

When $n \equiv 0 \pmod 3$, define

$$B_k = \{(x, y) : y \equiv 3x + i \pmod n, \ i = 0, 1, 2 \text{ and } \frac{(k-1)n}{3} \le x \le \frac{kn - 3}{3}\},$$

where $k = 1, 2, 3$. Clearly $H_k = \langle B_k \rangle$ and H_k is an arc induced and arc disjoint subdigraphs of $G_B(n, 3)$ for $k = 1, 2, 3$. Define a bijection $f : V(D) \cup A(D)$ to $\{1, 2, \ldots, p + q\}$ by $f(v) = 1 + v$, for all $v \in V(D)$. By

Theorem 1.12, we can construct a magic rectangle of order $3 \times n$ say $M_{3 \times n}$ with magic row sum $\dfrac{n(3n+1)}{2}$. Since *V*-super loop-rooted tree-magic labeling allows integer from $n+1$ to $4n$ to label the arcs, we add n to each entries of the magic rectangle $M_{3 \times n}$. Label the arcs of H_1, H_2, H_3 by using the entries in row1, row2 and row3 respectively of the magic rectangle. Therefore $G_B(n, 3)$ is a *V*-supermagic loop-rooted tree-decomposable digraph with magic constant $k = \dfrac{n(n+1)}{2} + \dfrac{n(3n+1)}{2} + n^2 = 3n^2 + n$.

When $n \equiv 1 \pmod{n}$, define

$$B_1 = \{(x, y) : y \equiv 3x + i \pmod{n},\ i = 0, 1, 2 \text{ and } 0 \le x \le \tfrac{n-1}{3}\}$$

$$\cup \left\{(x, y) : y \equiv 3x + i \pmod{n},\ x = \frac{n-1}{3},\ i = 0\right\};$$

$$B_2 = \{(x, y) : y \equiv 3x + i \pmod{n},\ i = 0, 1, 2 \text{ and } \tfrac{n-1}{3} + 1 \le x \le \tfrac{2(n-1)}{3} - 1\}$$

$$\cup \left\{(x, y) : y \equiv 3x + i \pmod{n},\ x = \frac{n-1}{3},\ i = 1, 2\right\}$$

$$\cup \left\{(x, y) : y \equiv 3x + i \pmod{n},\ x = \frac{2(n-1)}{3},\ i = 0, 1\right\};$$

$$B_3 = \{(x, y) : y \equiv 3x + i \pmod{n},\ i = 0, 1, 2 \text{ and } \tfrac{2(n-1)}{3} + 1 \le x \le n - 1\}$$

$$\cup \left\{(x, y) : y \equiv 3x + i \pmod{n},\ x = \frac{2(n-1)}{3},\ i = 2\right\}.$$

Clearly $H_k = \langle B_k \rangle$ and H_k is an arc induced and arc disjoint subdigraphs of $G_B(n, 3)$, $k = 1, 2, 3$. Define a bijection $f : V(D) \cup A(D)$ to $\{1, 2, \ldots, p + q\}$ by $f(v) = 1 + v$, for all $v \in V(D)$. By Theorem 1.12, we can construct a magic rectangle of order $3 \times n$ as $M_{3 \times n}$ with magic row sum $\dfrac{n(3n+1)}{2}$. Since *V*-super loop-rooted tree-magic labeling allows integer from

$n + 1$ to $4n$ to label the arcs, we add n to each entries of the magic rectangle $M_{3 \times n}$. Label the arcs of H_1, H_2, H_3 by using the entries in row1, row2 and row3 respectively of the magic rectangle. Therefore $G_B(n, 3)$ is a V-supermagic loop-rooted tree-decomposable digraph with magic constant $k = \dfrac{n(n+1)}{2}$ $+ \dfrac{n(3n+1)}{2} + n^2 = 3n^2 + n.$

When $n \equiv 2 \,(\text{mod } 3)$, define

$$B_1 = \{(x, y) : y \equiv 3x + i (\text{mod } n),\ i = 0, 1, 2 \text{ and } 0 \le x \le \frac{n-2}{3} - 1\}$$

$$\cup \left\{(x, y) : y \equiv 3x + i (\text{mod } n),\ x = \frac{n-2}{3},\ i = 0, 1\right\};$$

$$B_2 = \{(x, y) : y \equiv 3x + i (\text{mod } n),\ i = 0, 1, 2 \text{ and } \frac{n-2}{3} + 1 \le x \le \frac{2(n-2)}{3}\}$$

$$\cup \left\{(x, y) : y \equiv 3x + i (\text{mod } n),\ x = \frac{n-2}{3},\ i = 2\right\}$$

$$\cup \left\{(x, y) : y \equiv 3x + i (\text{mod } n),\ x = \frac{2(n-2)}{3} + 1,\ i = 0\right\};$$

$$B_3 = \{(x, y) : y \equiv 3x + i (\text{mod } n),\ i = 0, 1, 2 \text{ and } \frac{2(n-2)}{3} + 2 \le x \le n - 1\}$$

$$\cup \left\{(x, y) : y \equiv 3x + i (\text{mod } n),\ x = \frac{2(n-2)}{3} + 1,\ i = 1, 2\right\}.$$

Clearly $H_k = \langle B_k \rangle$ and H_k is an arc induced and arc disjoint subdigraphs of $G_B(n, 3)$ for $k = 1, 2, 3$. Define a bijection $f : V(D) \cup A(D)$ to $\{1, 2, \ldots, p + q\}$ by $f(v) = 1 + v,$ for all $v \in V(D)$. By Theorem 1.12, we can construct a magic rectangle of order $3 \times n$ say $M_{3 \times n}$ with magic row sum $\dfrac{n(3n+1)}{2}$. Since V-super loop-rooted tree-magic labeling allows integer from $n + 1$ to $4n$ to label the arcs, we add n to each entries of the magic rectangle $M_{3 \times n}$. Label the arcs of H_1, H_2, H_3 by using the entries in row1, row2 and row3 respectively of the magic rectangle. Therefore $G_B(n, 3)$ is a V-supermagic loop-rooted tree-decomposable digraph with magic constant $k = \dfrac{n(n+1)}{2} + \dfrac{n(3n+1)}{2} + n^2 = 3n^2 + n.$ □

Example 4.2. We illustrate the factor decomposition of generalized de Bruijn digraphs into loop-rooted trees in Figure 2.
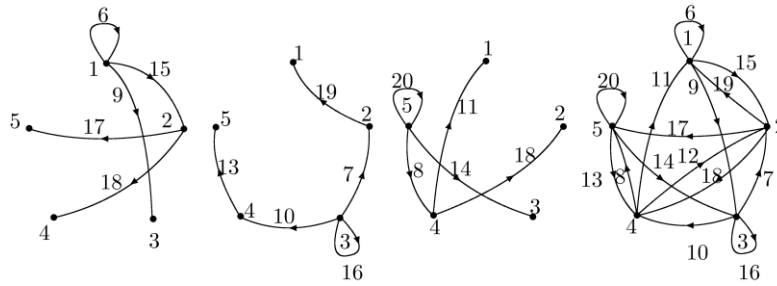

Figure 2. An example of *V*-super magic loop-rooted tree-decomposable digraph.

It is important to note that Theorem 4.1 is not a consequence of the following theorem. We consider $G_B(n, d)$ for the special case $d \mid n$.

Theorem 4.3. *Suppose $d \mid n$ and $(d-1) \mid (n-1)$. Then the digraph $G_B(n, d)$ is V-supermagic loop-rooted tree-decomposable.*

Proof. Since $(d-1) \mid (n-1)$ and $d \mid n$, then either n and d are even or n and d are odd. Define $B_k = \{(x, y) : y \equiv dx + i \pmod{n}, i = 0, 1, \ldots, d-1$ and $\frac{(k-1)n}{d} \leq x \leq \frac{kn}{d} - 1\}$. Clearly $H_k = \langle B_k \rangle$ and H_k is an arc induced and arc disjoint subdigraphs of $G_B(n, d)$ for $k = 1, 2, 3$. Define a bijection $f : V(D) \bigcup A(D)$ to $\{1, 2, \ldots, p+q\}$ by $f(v) = 1 + v$, for all $v \in V(D)$. If n is even, label the arcs of $H_1, H_2, H_3, \ldots, H_d$ by using the entries in row 1, row 2, $\ldots$, row d respectively using Table 3. If n is odd, construct a magic rectangle $d \times n$ by using Theorem 1.11 as $M_{d \times n}$. Since *V*-super loop-rooted tree-magic labeling allows integer from $n+1$ to $4n$ to label the arcs, we add n to each intries of the magic rectangle $M_{d \times n}$. Label the arcs of $H_1, H_2, H_3, \ldots, H_d$ by using the entries in row 1, row 2, $\ldots$, row d respectively of the magic rectangle. Therefore $G_B(n, d)$ is a *V*-supermagic loop-rooted tree-decomposable digraph magic constant $k = \dfrac{(nd+n)(nd+n+1)}{2d}$

$+ \dfrac{2n^2 d + n^2 + n}{2}$.   □

Table 3. Arc labels for arcs in H_k (n is even).

H_k	Arc labels for arcs in H_k						
H_1	$2n$	$2n+1$	$4n$	$4n+1$	...	nd	$n(d+1)$
H_2	$2n-1$	$2n+2$	$4n-1$	$4n+2$	...	$nd-1$	$nd+2$
H_3	$2n-2$	$2n+3$	$4n-2$	$4n+3$	...	$nd-2$	$nd+3$
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
H_{d-1}	$n+2$	$3n-1$	$3n+2$	$5n-1$	...	$n(d-1)+2$	$nd+n-1$
H_d	$n+1$	$3n$	$3n+1$	$5n$	...	$n(d-1)+1$	$nd+1$

Theorem 4.4. *Suppose $d \mid n$ and $\gcd(n, d-1)=1$. Then the digraph $G_B(n, d)$ is V-supermagic loop-rooted tree-decomposable.*

Proof. The arc set of $G_B(n, d)$ can be decomposed as $A(G_B(n, d)) = \bigcup_{k=1}^{d} B_k$, where $B_k = \{(x, y) : y \equiv dx + i \pmod{n}, i = 0, 1, \ldots, d-1$ and $\frac{(k-1)n}{d} \le x \le \frac{kn}{d} - 1\}$. Clearly $H_k = \langle B_k \rangle$ and H_k is an arc induced and arc disjoint subdigraphs of $G_B(n, d)$ for $k = 1, 2, 3$. By a similar arguement as in Theorem 4.3, we can prove that $G_B(n, d)$ is a V-supermagic loop-rooted tree-decomposable digraph.   □

When d does not divide n, we now consider the special case $n = dk \pm 1$. for some integer $k > 0$.

Theorem 4.5. *Suppose $n = dk-1$ or $n = dk+1$, $k > 0$ and either $\gcd(n, d-1)=1$ or $(d-1) \mid (n-1)$. Then the digraph $G_B(n, d)$ is a V-supermagic loop-rooted tree-decomposable, if any one of the following conditions hold.*

(i) *n is even.*

(ii) *Both n and d are odd.*

Proof. When $n = dk-1$. Define $S_m = A_m + B_m + C_m$, $m = 1, 2, \ldots, d$. $A_d = \phi$ and $C_1 = \phi$.

$$A_m = \bigcup_{i=0}^{d-1-m} \{(x, y) : y \equiv dx + i \pmod{n},\ x = mk - 1\};$$

$$B_m = \bigcup_{i=0}^{d-1} \{(x, y) : y \equiv dx + i \pmod{n}\,(m-1)k \le x \le mk - 2\};$$

$$C_m = \bigcup_{i=d-m+1}^{d-1} \{(x, y) : y \equiv dx + i \pmod{n},\ x = (m-1)k - 1\}.$$

Clearly $H_m = \langle S_m \rangle$ and H_m is an arc induced and arc disjoint subdigraphs of $G_B(n, d)$, for $m = 1, 2, \ldots, d$. By similar arguement as in Theorem 4.3, we can prove $G_B(n, d)$ is a *V*-supermagic loop-rooted tree-decomposable digraph.

When $n = dk + 1$. Define A_m, B_m and C_m, $m = 1, 2, \ldots, d$ as follows:

$$A_m = \bigcup_{i=m-1}^{d-1} \{(x, y) : y \equiv dx + i \pmod{n},\ x = k(m-1)\};$$

$$B_m = \bigcup_{i=0}^{d-1} \{(x, y) : y \equiv dx + i \pmod{n}\,k(m-1) + 1 \le x \le mk - 1\};$$

$$C_m = \bigcup_{i=0}^{m-1} \{(x, y) : y \equiv dx + i \pmod{n},\ x = km\}.$$

Clearly $H_m = \langle A_m \cup B_m \cup C_m \rangle$ and H_m is an arc induced and arc disjoint subdigraphs of $G_B(n, d)$, for $m = 1, 2, \ldots, d$. By a similar arguement as in Theorem 4.3, we can prove $G_B(n, d)$ is a *V*-supermagic loop-rooted tree-decomposable digraph. □

Theorem 4.6. *If $G_B(n, d)$ can be decomposed into d loop-rooted trees, then $G_B(n, d)$ is not a V-supermagic loop-rooted tree-decomposable digraph, when n is odd and d is even.*

Proof. For a generalized de Bruijn digraph $p = n$ and $q = nd$. By Theorem 3.1., we have

$$k = \frac{p(p+1)}{2} + \frac{q^2 + 2pq + q}{2h}$$

$$= \frac{n(n+1)}{2} + \frac{n^2 d^2}{2d} + \frac{2n(nd)}{2d} + \frac{nd}{2d}$$

$$= \frac{n(n+1)}{2} + \frac{n^2 d}{2} + n^2 + \frac{n}{2}.$$

The first and second term in the above equation is an integer but the fourth term is not an integer and hence the result follows.   □

5. Conclusion and Scope

In this paper, we have introduced the concept of *V*-super vertex magic (out-magic) labeling in digraphs and examined the *V*-super vertex magic (out-magic) labeling in generalized de Bruijn digraphs. Similarly we can study the properties of *V*-super vertex in-magic labeling in digraphs. The *V*-super vertex out-magic labeling and *V*-super vertex in-magic labeling are one and the same for regular digraphs. Next, we introduced *V*-super *H*-magic labeling of digraphs and found some *V*-supermagic loop-rooted tree-decomposable digraphs. There are some other classes $(n = dk \pm 2, n = dk \pm 3)$ of generalized de Bruijn digraph which admits *V*-super loop-rooted tree-magic labeling. We conclude this paper with the following open problems.

Open Problem 1. Find all *V*-supermagic loop-rooted tree-decomposable generalized de Bruijn digraphs, when $n = dk \pm r$, $r = 0, 1, 2, \ldots, d - 1$.

Open Problem 2. Characterize all *V*-supermagic loop-rooted tree-decomposable digraphs.

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